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of leisure occur. I have acted under the impression of the future value of such a work in my own Office, and have just completed the classification of the policies from the beginning; and it is satisfactory to state, that the results, after additions and deductions for 20 years, exactly agree with the particulars of the policies in force, though in the course of that time many minute alterations of premiums, ages, &c., must have occurred. Those who feel the value of time will not despise even slight suggestions, such as I have ventured to lay before you. If they help to remove the serious difficulty to combining the experience of several Assurance Companies—viz., the labour and time which cannot be spared for bringing the first materials together—they will, I am convinced, lead to results hereafter which not only the actuary but the public will benefit by. Simplicity is often the effect of experience; and whilst other animals may by instinct work to perfection, man must improve by the communicated knowledge of another, and profit even by his mistakes and failures. It is this feeling which has induced me to bring before you a plan which I have found by experience to be useful, which is simple in arrangement, capable of being used for other official purposes, and which, with very little addition to present labour, may afford the means hereafter of making new discoveries in the science of probability and the statistics of life.

## *On the Method of Approximating to the Values of Deferred and other Life Annuities, when payable Half Yearly and Quarterly.*

*By* HOLMES IVORY, *one of the Vice-Presidents of the Institute of Actuaries.*

[Read before the Institute 24 April, 1854, and ordered by the Council to be printed.]

THE difference between the value of an immediate life annuity when payable yearly, and that of the same annuity when payable half yearly or quarterly, has, as is well known, been variously estimated by our leading authorities.

According to Dr. Price and Mr. Morgan, "a fifth of a year's purchase will be generally more than a sufficient addition if the value of the annuity is desired payable half yearly, and three tenths of a year's purchase if the value of the annuity is desired payable quarterly." * Mr. Baily, again, following Simpson, lays it down as a rule "that the value of annuities payable *yearly* must be increased nearly one fourth of a year's purchase in order to show the value of the same annuities payable *half yearly,* and that they must be increased nearly three eighths of a year's purchase in order to show the value of the same annuities payable *quarterly.*" The same rule is also given by Mr. Milne.†

The algebraical investigations from which the above rules were deduced are fully detailed by the authors referred to; and the difference between the results arrived at is sufficiently explained by the fact, that Dr. Price has proceeded on the assumption of interest being received and accumulated half yearly and quarterly at the same periods of the year at which the annuities are payable, and at rates equal to *one half* and *one fourth* respectively of the nominal annual rate, whereas Simpson and Baily take the interest for portions of a year at such a rate as will produce the nominal annual rate and no more at the end of the year. Which of the two rules ought to be preferred may perhaps admit of some difference of opinion; for though that given by Baily is now almost universally adopted, the views of Dr. Price as to what is implied in speaking of a certain rate of interest payable half yearly or quarterly seem most in accordance with general use and understanding. In evidence of this it may be sufficient to refer to the fact, that dividends or interest on Government and Bank Stocks, though spoken of as yielding 3, 5, 10, or other per cent. premium, are all payable half yearly, at rates just equal to half the annual interest.

But without dwelling farther on this point, it is proposed in what follows to adopt Mr. Baily's rule as the basis of calculation. It is at the same time of importance to remark, that whichever rule may be adopted, the specific addition thereby required, in respect of the half yearly and quarterly payments of the annuity, remains the same, whatever the rate of interest, the age of the party, or the number of lives concerned, provided only that the annuities are immediate and to continue to the close of life. The closeness of the approximation thus obtained, and the slight effect which the age of the annuitant has on the result, will be at once appreciated when it is stated that, while the approximate value of an annuity payable half yearly, to a person aged 15, amounts by Mr. Baily's rule to (interest 3 per cent.) £22·8320, the exact value by calculation on the footing of interest being received half yearly, at a rate to produce 3 per cent. per annum, is £22·8296, being a difference of only £·0024;* and that at age 85 the approximate value of an annuity payable half yearly is, according to Mr. Baily's rule, £3·2830, the exact calculated value on the footing of interest payable half yearly, at 3 per cent. per annum, being £3·2795, showing a difference of £·0035.

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* Price, vol. i. p. 246, 6th and 7th Editions.

† Baily, ch. x. § 355. Milne, ch. viii. § 485.

When the annuities are not immediate, but temporary, deferred, or contingent, the rule must be modified to suit the special circumstances of each case; and the object of the present communication is to endeavour to determine the nature and extent of the requisite modifications.

Mr. Sang, it is believed, is the only authority who has especially adverted to this particular point; and the rules given by him in the Introduction to his *Tables* have reference only to deferred and temporary annuities on single lives, and require to be adjusted to suit the ordinary form of annuity tables.† The whole question, indeed, is not in itself of much commercial importance; still, as accuracy is at all times desirable, and as questions do occasionally occur in which the effect of half yearly and quarterly payments have to be considered and allowed for, it is hoped that the following suggestions may not be without some interest to the practical as well as to the theoretical actuary.

## I. DEFERRED ANNUITIES.

The value of a yearly annuity, deferred $n$ years on a life now aged $m$ years, is found by calculating the value at age $m$ of the sum which at age $m+n$ will purchase an annuity to a life aged

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[*] The formula for finding the value of an annuity, $a$, payable half yearly, as deduced by Mr. Baily (ch. x. § 352), is

$$\frac{2(1+\rho)^{\frac{1}{2}}+2+\rho}{4(1+\rho)^{\frac{1}{2}}}+a\times\frac{1}{4(1+\rho)^{\frac{1}{2}}};$$

which expression, since the quantity $\frac{2(1+\rho)^{\frac{1}{2}}+2+\rho}{4(1+\rho)^{\frac{1}{2}}}$ seldom much exceeds unity, may be taken without material error as equal to $a+\frac{1}{4(1+\rho)^{\frac{1}{2}}}$; and $\frac{1}{4(1+\rho)^{\frac{1}{2}}}$ being seldom much below $\frac{1}{4}$, the expression may be still farther reduced to $a+\frac{1}{4}$; and hence the rule given by Mr. Baily. When interest is at 3 per cent., the quantity $\frac{2(1+\rho)^{\frac{1}{2}}+2+\rho}{4(1+\rho)^{\frac{1}{2}}}$ is equal to 1·000054, and the quantity $\frac{1}{4(1+\rho)^{\frac{1}{2}}}$ is equal to ·2463323, or ·0036677 less than $\frac{1}{4}$.

[†] Mr. Sang has framed his *Tables* on the assumption of the annuity being payable in advance or at the beginning of the year. The tables usually made use of, however, assume that the annuity is payable at the end of the year. Hence the value of an immediate annuity, as given by Mr. Sang, is always greater by 1 than that given in the ordinary tables.

$m+n$;* the formula according to the usual notation being

$$a_{m/n} = \frac{l_{m+n}v^n}{l_m} \times a_{m+n}$$

To find the value of the same annuity payable *half yearly*, it is obvious that the approximate addition of £·25, or one fourth of a year's purchase, in respect of the half yearly payments, must be made to the value of the annuity at an advanced age, or to $a_{m+n}$, and the present value of the sum thus obtained found as at the age $m$. Hence the following formula:—

$$a_{m/n} = \frac{l_{m+n}v^n}{l_m} \times (a_{m+n} + {\cdot}25) = a_{m+n} \times \frac{l_{m+n}v^n}{l_m} + {\cdot}25\,\frac{l_{m+n}v^n}{l_m};$$

or, by the D and N columns,

$$\frac{4\,\mathrm{N}_{m+n} + \mathrm{D}_{m+n}}{4\,\mathrm{D}_m}.$$

In like manner, the value of the same annuity, payable *quarterly*, will be obtained by the following formula:—

$$(a_{m+n} + {\cdot}375) \times \frac{l_{m+n}v^n}{l_m} = a_{m+n} \times \frac{l_{m+n}v^n}{l_m} + {\cdot}375\,\frac{l_{m+n}v^n}{l_m};$$

or, by the D and N columns,

$$\frac{8\,\mathrm{N}_{m+n} + 3\,\mathrm{D}_{m+n}}{8\,\mathrm{D}_m}.$$

Expressed in words, the rule may be stated shortly as follows:—

1. *For an annuity payable half yearly.*—To the value of a deferred annuity payable yearly, add the present value of a quarter of a year's purchase of the annuity payable at the end of the deferred period.

2. *For an annuity payable quarterly.*—To the value of a deferred annuity payable annually, add the present value of three eighths of a year's purchase of the annuity payable at the end of the deferred period.

The same results will be obtained according to Mr. Sang's method, when modified to suit the ordinary form of annuity tables. Thus $a_{m/n}$ being the value of a yearly annuity of £1 deferred $n$ years, and $a_{m/n-1}$ the value of the same annuity deferred for $n-1$ years, it may be assumed that $\frac{a_{m/n} + a_{m/n-1}}{2}$ will represent very nearly the value of an annuity of £1 deferred $n - \frac{1}{2}$ years, or one of which the first payment is to be made at the end of the first half year after the expiry of $n$ years; and consequently, that $\frac{a_{m/n} + a_{m/n-1}}{4}$ will represent the value of an annuity of 10*s.* per annum, of which the first payment is to be made six months after the expiry of $n$ years. But $\frac{a_{m/n}}{2}$ being the value of an annuity of 10*s.*, of which the first payment is to be made at the end of a year after the expiry of $n$ years, it follows that the sum of these two expressions will give very nearly the value of an annuity of £1 payable by half yearly instalments of 10*s.* each; and accordingly,

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* In conformity with general practice, an annuity deferred $n$ years is here held to mean an annuity to be *entered* on at the end of $n$ years, and of which the first payment is to be made at the age $m+n+1$. This mode of distinguishing the period of entering on an annuity from that on which the payment is to commence is apt to lead to mistakes; and it might be well, therefore, were the suggestion made by Mr. Farr in his *Treatise on the Finance of Life Assurance* (p. 30), appended to the octavo edition of the *Twelfth Annual Report of the Registrar General*, to be generally adopted, and an annuity deferred $n$ years held to be one of which the first payment is to be made at the age $m+n$.

$$\frac{a_{m/n}}{2} + \frac{a_{m/n} + a_{m/n-1}}{4} = \frac{3a_{m/n} + a_{m/n-1}}{4}$$

will be the value of a deferred annuity $a_{m/n}$. payable half yearly.

When the annuity is payable quarterly, the value will in like manner be found as follows:—As $\frac{a_{m/n} + a_{m/n-1}}{2}$ gives, as above shown, the value very nearly of an annuity of £1 of which the first payment is to be made six months after the expiry of $n$ years, so $\frac{a_{m/n-1} + \frac{a_{m/n} + a_{m/n-1}}{2}}{2}$ will be the value of an annuity of £1 of which the first payment is to be made at the end of three months, or one fourth of a year after the expiry of $n$ years; and $\frac{a_{m/n} + \frac{a_{m/n} + a_{m/n-1}}{2}}{2}$ will be the value of an annuity of £1 of which the first payment is to be made at the end of three quarters of a year after the expiry of $n$ years. But $a_{m/n}$ being the value of an annuity of which the first payment is to be made at the end of one year after the expiry of $n$ years, it follows that the sum of these four expressions will give the value of four several annuities of £1 each of which the first payments are to be made at the end respectively of one quarter, one half, three quarters, and one whole year after expiry of $n$ years, and will be expressed by

$$\frac{10a_{m/n} + 6a_{m/n-1}}{4} = \frac{5a_{m/n} + 3a_{m/n-1}}{2};$$

and hence, dividing by 4, we get the value of an annuity of £1 payable quarterly, or $\frac{5a_{m/n} + 3a_{m/n-1}}{8}$

The following examples in numbers will show the practical working of the rules, and the accordance of the results according to both methods:—

(1.) Required the value of an annuity of £1 deferred 22 years, and payable half yearly to a person now aged 35.

*First Method.*

| | £ |
|---|---|
| Value of an annuity deferred 22 years on a life of 35, and payable *yearly* | 4·43587 |
| Add value of £·25 payable to a life now 35, at end of 22 years | ·09548 |
| Value of deferred annuity payable half yearly | £4·53135 |

*Second Method.*

| | £ |
|---|---|
| Value of an annuity deferred 22 years on a life now 35, payable yearly | 4·43587 |
| | 3 |
| | 13·30761 |
| Value of ditto, deferred 21 years | 4·81780 |
| | 4) 18·12541 |
| Value of deferred annuity payable half yearly | £4·53135 |

(2.) Required the value of the same annuity payable quarterly.

*First Method.*

| | £ |
|---|---|
| Value of deferred annuity on life 35, as above | 4·43587 |
| Value of three eighths, or £·375, payable at end of 22 years | ·14322 |
| Value of deferred annuity payable quarterly | £4·57909 |

*Second Method.*

| | £ |
|---|---|
| Value of annuity payable yearly, as above | 4·43587 |
| | 5 |
| | 22·17935 |
| Ditto of ditto, deferred 21 years, £4·81780 × 3 | 14·45340 |
| | 8) 36·63275 |
| Value of deferred annuity payable quarterly | £4·57909 |

## II. TEMPORARY ANNUITIES.

The value of a temporary annuity payable annually being equal to the present value of an immediate annuity on the whole life, less the present value of an annuity deferred for the period the temporary annuity is to continue, it follows, from what has been already shown, that the value of the same annuity payable half yearly will be found by the following formula:—

$$(a_m+\cdot25)-\left(\frac{l_{m+n}v^n}{l_m}\times\overline{a_{m+n}+\cdot25}\right)=a_m-a_{m_n|}+\cdot25\times\left(1-\frac{l_{m+n}v^n}{l_m}\right)$$

$$=a_{m_n|}+\cdot25\times\left(1-\frac{l_{m+n}v^n}{l_m}\right).$$

For the same annuity payable quarterly the formula will be

$$a_{m_n|}+\cdot375\times\left(1-\frac{l_{m+n}v^n}{l_m}\right).$$

Hence the following rule:—To the value of a temporary annuity payable annually, add one fourth or three eighths of a year's purchase, according as the value is wanted payable half yearly or quarterly, less the value of the same sum payable at the end of the period for which the temporary annuity is to continue.

According to Mr. Sang's method, the value will be found as follows:—

$a_{m_n|}$ being the value of a yearly annuity for $n$ years on a life aged $m$, and payable at the end of the year, $a_{m_{n-1}|}+1$ will represent the value of an annuity on same life for $n$ years payable at the beginning of the year, and $\dfrac{a_{m_n|}+a_{m_{n-1}|}+1}{2}$ will be the value very nearly of an annuity of £1 of which the first payment is to be made six months hence; consequently, $\dfrac{a_{m_n|}+a_{m_{n-1}|}+1}{4}$ will give the value of an annuity of 10*s.* of which the first payment is to be made six months hence. But $\dfrac{a_{m_n|}}{2}$ being the value of a temporary annuity of 10*s.* per annum of which the first payment is to be made at the end of a year, it follows that

$$\frac{a_{m_n|}}{2}+\frac{a_{m_n|}+a_{m_{n-1}|}+1}{4}=\frac{3a_{m_n|}+a_{m_{n-1}|}+1}{4}$$

will represent very nearly the value of a temporary annuity of £1 payable *half yearly.*

In like manner, the value of a temporary annuity payable *quarterly* will be found to be

$$\frac{5a_{m_n|}+3\left(a_{m_{n-1}|}+1\right)}{8}.$$

The following example shows the accordance of the results obtained by the two methods.

(1.) According to the *first method,* the value of a temporary annuity of £1 for 22 years payable half yearly to a person now aged 35, is found as follows :—

| | | |
|---|---|---|
| Value of temporary annuity of £1 for 22 years on life 35, payable yearly | | £13·9978 |
| Add for half yearly payments | £·2500 | |
| Less present value of £·25, payable at end of 22 years | ·0955 | |
| | | ·1545 |
| Value of temporary annuity payable half yearly | | £14·1523 |

(2.) Taking the second method, the calculation will stand thus :—

| | |
|---|---|
| Value of temporary annuity for 22 years on life 35, payable yearly | £13·9978 |
| | 3 |
| | 41·9934 |
| Do. of temporary annuity for 21 years, payable yearly (£13·615) + 1 | 14·6159 |
| | 4)56·6093 |
| Value of temporary annuity payable half yearly, as above | £14·1523 |

## III. Reversionary Annuities.

Since the present value of a reversionary annuity is evidently the same as the present value of the sum which would be sufficient to provide for such annuity at the close of the year in which the life on whose failure the opening of the reversion depends may drop—and seeing further that the first payment of the reversionary annuity falls, from the nature of the case, to be made at the same period—it follows that the annuity must be dealt with as one payable in advance; and consequently, to convert the present value of a reversionary annuity payable yearly into that of one payable half yearly, we must, in respect that only one half in place of a whole year's annuity will be paid in advance, deduct in the first place, from the value of the annuity when payable yearly, the present value of £·5 or half a year's purchase payable as at the close of the year in which the life may drop, and then add the present value of the usual addition of £·25 or one fourth of a year's purchase, payable as at same period, in respect of the half yearly payments: or, which is the same thing, we must deduct from the present value of the reversionary annuity payable yearly the value of one fourth of a year's purchase payable as at the end of the year in which the life may fail; and, adopting Mr. Jones' notation, the formula will be

$$a_{m_1} - a_{m_1.m} - \frac{A_{m_{(1)}m_1}}{4}$$

where $a_{m_1}$ represents the surviving life, and $A_{m_{(1)}m_1}$ the value of £1 to be received at the end of the year in which the life aged $m$ may fail, provided that life be survived by another aged $m_1$.

For example: the value of a reversionary annuity of £1, payable *half yearly* to a person now aged 34, after the death of another now aged 45, will be found as follows:—*

| | |
|---|---|
| The value of the reversionary annuity payable annually being £18·675—£13·409 (Jones' Carlisle Table) . . . | £5·266 |
| Deduct the value of £·25 (£·5 – £·25) payable to 34 after death of 45 (Gray's Tables) . . . . . . . . | ·092 |
| Value of reversionary annuity payable half yearly . | £5·174 |

The value of a reversionary annuity payable *quarterly* is found on the same principle, by deducting from the value payable yearly the present value of £·75 less £·375(=£·375), payable at the end of the year in which the life may drop,=

$$a_{m_1} - a_{m_1.m} - ·375 \times A_{m_{(1)}.m_1}$$

## IV. SURVIVORSHIP ANNUITIES.

These will be dealt with in a similar manner; and their value, when payable half yearly, will be found by adding to the value when payable yearly the present value of £·5 – £·25 (=£·25) payable at death of either, the formula being

$$a_m + a_{m_1} - 2a_{m.m_1} + \frac{A_{(m m_1)}}{4}$$

---

* The *rationale* of the operation will perhaps be better exhibited in the following form:—

The present value of a reversionary annuity on a life 34 after one of 45 being £5·266, is equivalent to a *sum* payable at the end of the year in which 45 dies (leaving 34 alive) of £14·310, which represents the value as at that date of an annuity of £1 payable annually during the remainder of the life of a person now aged 34, after the death of another aged 45, and of which the first payment has to be made in advance or at the end of the year in which 45 dies. But as the annuity is to be paid by half yearly instalments, the first payment will only be £·5 in place of £1, and we must therefore deduct £·500; on the other hand, there has to be added in respect of the half yearly payment £·250, leaving to be deducted £·250; and the value of an annuity of £1 payable half yearly to a life now aged 34, ascertained as at the end of the year in which 45 dies, is £14·060, the present value of which at age 34 is £5·174, agreeing with the result in the text.

For example: the value of an annuity of £1 payable to the survivor of 34 and 45 will be

| | £ |
|---|---|
| 18·675 + 15·863—(2 × 13·409) . . . . . . . | 7·720 |
| *Plus* the value of £0·25 payable at death of either $\frac{A(34{\cdot}45)}{4}$ . | ·145 |
| Value of survivorship annuity payable half yearly . | £7·865 |

The above cases comprise most of those usually met with in practice; and as the same principles will regulate the calculation in the other and less common forms of life annuities, it seems unnecessary to add to the length of the present communication by going into farther details in regard to them.

## I. *On the First Parliamentary Committee on Insurance; with Remarks illustrative of other facts connected with the History of Insurance.*

## II. *A Review of some Recommendations of the Select Committee of the House of Commons on Assurance Associations,* 1853.

*By* Frederick Hendriks, *Actuary to the Globe Insurance Company.*

[I., concluded from page 131; II., *vide* page 324, *post.*]

THE preceding were the last two petitions which had to be reported on. The petitioners for the Forfeited Estates project were advised, but not at the right time, that their existing Act gave them the powers they were petitioning for. Upon this advice they offered to waive further proceedings; but this was disallowed, as the Attorney General elicited, in his examination of Case Billingsley, that the petition had been delivered at the Council Office without the privity of a great number of those who were mentioned in it, and that Mr. Billingsley had the authority of a small number of them only, to desire to withdraw it. The substance of the Attorney General's final judgment was, that the petitioners had by their Act power to purchase lands, but that such power was limited to the purpose for which the Corporation was erected; and that the transaction which had taken place in respect of the subscription for granting annuities and life assurances was foreign to the ends of the incorporation, of a "very dangerous tendency, and highly in derogation of the royal prerogative"; and for which reason the Corporation was liable to prosecution.